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On Spherical t-designs in $\mathbb{R}^2$

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Let X be a spherical t-design in $\mathbb{R}^2$ with $|X| = n$. It is shown that:
(i) for $t+1 \leqslant n \leqslant 2t+1$, X must be a regular n-gon;
(ii) for $n = 2t+2$, X is the union of two regular $(t+1)$-gons;
(iii) for each $n \geqslant 2t+3$, there are many $(\aleph_1)$ X's which can not be decomposed into the union of regular k_i-gons where $k_i \geqslant (t+1)$.

1. INTRODUCTION

Spherical t-designs in real Euclidean space $\mathbb{R}^d$ were defined and studied by Delsarte, Goethals and Seidel [1]. A spherical t-design X is a finite non-empty set of unit vectors in the Euclidean space $\mathbb{R}^d$ such that, for $k = 1, 2, \ldots, t$, the kth moments of X are constants with respect to orthogonal transformations of $\mathbb{R}^d$, or equivalently, that $\sum_{x \in X} h(x) = 0$ for all homogeneous harmonic polynomials $h(x)$ on $\mathbb{R}^d$ of degree $1, 2, \ldots, t$.* (See [1, Definition 5.1 and Theorem 5.2]). Such designs have lower bounds for their sizes [1, Theorems 5.11 and 5.12]. In particular, for a spherical t-design X in $\mathbb{R}^2$, we have $|X| \geqslant t+1$ and $|X| = t+1$ iff X is the set of vertices of a regular $(t+1)$-gon [1, Example 5.14]. For convenience, we will simply call the vertex set of a regular n-gon a regular n-gon and call a spherical t-design in $\mathbb{R}^2$ a t-design. Since a t-design is a $(t-1)$-design and the union of t-designs is again a t-design, we have that the union of regular k_i-gons $(k_i \geqslant t+1)$ is a t-design. This gives a great supply of t-designs. We define the following:

A t-design X is a *group-type* t-design iff $X = \cup_{i=1}^{s} G_i$ where G_i are regular k_i-gons with $k_i \geqslant t+1$ and $\sum_{i=1}^{s} k_i = |X|$.

A t-design is a *non-group-type* t-design if it is not a group-type t-design.†

In this note, the author will investigate spherical t-designs in $\mathbb{R}^2$ with the help of the complex variable and prove the following theorem.

THEOREM A. *Let X be a spherical t-design in $\mathbb{R}^2$ with $|X| = n$. Then*

(i) *for $t+1 \leqslant n \leqslant 2t+2$, X is always a group-type t-design; more precisely, when $t+1 \leqslant n \leqslant 2t+1$, X is a regular n-gon and when $n = 2t+2$, X is the union of two regular $(t+1)$-gons;*‡

(ii) *for each $n \geqslant 2t+3$, besides group-type t-designs, there are many $(\aleph_1)$ non-group-type t-designs X's.*

2. PRELIMINARY LEMMAS AND THE CASE $n \leqslant 2t+2$

Let Harm(k) denote the linear space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^2$. It is known that $\dim(\mathrm{Harm}(k)) = 2$ for $k = 1, 2, \ldots$. (See, for example, [1, Theorem 3.2]). Let $z = x + iy$ denote the complex variable. Then $\mathrm{Re}(z^k)$ and $\mathrm{Im}(z^k)$ form a base for Harm(k). This is based on the facts that the real and imaginary parts of a holomorphic function are harmonic, that both $\mathrm{Re}(z^k)$ and $\mathrm{Im}(z^k)$ are homogeneous and of degree k, and that they are linearly independent. Thus the definition of a t-design X

* The definition in [1] requires that all the elements in X be distinct. Here, in our theorem, we do not need this requirement.

† It seems that all explicit examples of spherical t-designs appearing in the literature were group-type t-designs.

‡ Including the regular $2(t+1)$-gon.

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that requires $\sum_{x\in X} h(x)=0$ for all $h\in \mathrm{Harm}(k)$, $k=1,2,\ldots,t$, can be reformulated as follows:

LEMMA 1

$$X=\{z_1, z_2, \ldots, z_n\}\subset S^1=\{z\in\mathbb{C}\,|\,|z|=1\} \text{ is a } t\text{-design iff}$$

(i) $\sum_{i=1}^{n} z_i^k=0$ for $k=1,\ldots,t$, *iff*

(ii) $\sum_{1\le i_1<i_2<\cdots<i_k\le n} z_{i_1}z_{i_2}\cdots z_{i_k}=0$ *for* $k=1,2,\ldots,t$.

Condition (ii) of this lemma suggests that we consider the product $(z-z_1)(z-z_2)\cdots(z-z_n)$. By (ii), in the expansion of $(z-z_1)(z-z_2)\cdots(z-z_n)$, the first t terms following z^n will all be equal to zero if X is a t-design. Furthermore, the t terms preceding the constant term will also be equal to zero. This is shown in Lemma 2. In order to simplify coefficients in the expansion of $(z-z_1)\cdots(z-z_n)$, we give the following definitions.

DEFINITION. Given $X=\{z_1, z_2, \ldots, z_n\}\subset S^1$, define $X^*=\{z_1^*, z_2^*, \ldots, z_n^*\}$ by $z_k^*=z_k\,e^{-i\theta}(1\le k\le n)$ where we choose θ in such a way that $e^{in\theta}=z_1z_2\cdots z_n$ and $0\le n\theta<2\pi$.

Clearly, $z_1^*z_2^*\cdots z_n^*=1$ and X^* is a t-design iff X is a t-design. We identify two t-designs to be the same design if one is obtained from the other by a rotation.

DEFINITION. Given $X=\{z_1, z_2, \ldots, z_n\}\subset S^1$, define the polynomial $f_X(z)$ by

$$f_X(z)=(z-z_1^*)(z-z_2^*)\cdots(z-z_n^*).$$

LEMMA 2. *Let* $X=\{z_1, z_2, \ldots, z_n\}$ *be a subset of* S^1 *and let* $f_X(z)=z^n-a_{n-1}z^{n-1}+\cdots+(-1)^k a_{n-k}z^{n-k}+\cdots+(-1)^n a_0$. *Then*

(i) $a_0=1$,

(ii) $a_{n-k}=\overline{a_k}$ *for* $k=0,1,\ldots,n$,

(iii) *X is a t-design iff* $a_{n-1}=a_{n-2}=\cdots=a_{n-t}=0$ *and iff* $a_t=a_{t-1}=\cdots=a_1=0$.

PROOF

(i) $a_0=z_1^*z_2^*\cdots z_n^*=1$.

(ii) $$a_{n-k}=\sum_{1\le i_1\le i_2<\cdots<i_k\le n} z_{i_1}^*z_{i_2}^*\cdots z_{i_k}^*=\sum\frac{z_{i_1}^*z_{i_2}^*\cdots z_{i_k}^*}{z_1^*z_2^*\cdots z_n^*}$$

$$=\sum_{1\le j_1<j_2<\cdots<j_{n-k}\le n}\frac{1}{z_{j_1}^*z_{j_2}^*\cdots z_{j_{n-k}}^*}=\sum\overline{z_{j_1}^*z_{j_2}^*\cdots z_{j_{n-k}}^*}=\overline{a_k}.$$

(iii) follows from (ii) and Lemma 1.

By Lemma 2, if$X$ is a t-design, then

(i) when $t+1\le n\le 2t+1$, we have $f_X(z)=z^n+(-1)^n$ and

(ii) when $n=2t+2$, we have $f_X(z)=z^{2t+2}+(-1)^{t+1}a_{t+1}z^{t+1}+1=(z^{t+1}-A)(z^{t+1}-B)$ for some $A, B\in S^1$.

Therefore, we get the first assertion of Theorem A. That is the following corollary.

COROLLARY 2.1. *Let X be a t-design with $|X| = n$. Then*
(i) *X is a regular n-gon if $t+1 \leq n \leq 2t+1$,*
(ii) *X is the union of two regular $(t+1)$-gons if $n = 2t+2$.*

3. THE CASE $n = 2t+3$

As we have discussed in the Introduction, there are many group-type t-designs of any given size $\geq t+1$. In the following, we are going to see there are also many (i.e. $\aleph_1$) non-group-type t-designs of any size $\geq 2t+3$.

First, we start with the case $n = 2t+3$. Let X be a t-design with $|X| = n = 2t+3$. By Lemma 2, $f_X(z) = z^{2t+3} + b_k z^{t+2} - \overline{b_k} z^{t+1} - 1$ for some $b_k \in \mathbb{R}$. Conversely, if $f(z) = z^{2t+3} + b_k z^{t+2} - \overline{b_k} z^{t+1} - 1$ has all its zeros in S^1, then those zeros form a t-design of size $2t+3$. Suppose X is a group-type t-design. Then X must be either a regular $(2t+3)$-gon or the union of a regular $(t+2)$-gon and a regular $(t+1)$-gon. In this case $f_X(z)$ has its coefficient $b_k = 0$ or $|b_k| = 1$. Therefore, in order to show the existence of a non-group-type t-design of size $2t+3$, it is sufficient to show that there exists a complex number $b_k \neq 0$ with $|b_k| \neq 1$ such that $f(z) = z^{2t+3} + b_k z^{t+2} - \overline{b_k} z^{t+1} - 1$ has all its zeros in S^1.

Let us put $e^{i\theta} (0 \leq \theta < 2\pi)$ for z in the equation $z^{2t+3} + b_k z^{t+2} - \overline{b_k} z^{t+1} - 1 = 0$. Then we get $e^{i\theta(2t+3)} + b_k e^{i\theta(t+2)} = \overline{b_k e^{i\theta(t+1)} + 1}$. Multiplying both sides by $e^{-i\theta(t+3/2)}$, we get the equation $e^{i\theta(t+3/2)} + b_k e^{i\theta(1/2)} = \overline{e^{i\theta(t+3/2)} + e^{i\theta(1/2)}}$. Thus

$$\operatorname{Im}(e^{i\theta(t+3/2)} + b_k e^{i\theta(1/2)}) = 0, \text{ i.e.}$$

$$\sin(t+3/2)\theta + R \sin(1/2\theta + a) = 0 \text{ where } b_k = R e^{ia} (R, a \in \mathbb{R}).$$

Now our aim (i.e. to show the existence of non-group-type t-design) becomes to show that there exist some real numbers $R \neq 0$ or 1 and a such that the equation $\sin(t+3/2)\theta + R \sin(1/2\theta + a) = 0$ has $(2t+3)$ solutions for $\theta \in [0, 2\pi)$. Put $a = 0$. This equation becomes $\sin(t+3/2)\theta + R \sin(1/2)\theta = 0$ and $\theta = 0$ is a zero (for any R). Now, our aim becomes to get some $R \neq 0$ and 1 such that the equation $R = -\sin(t+3/2)\theta/\sin(\theta/2)$ has $(2t+2)$ solutions for $\theta \in (0, 2\pi)$. Let $R(\theta) = -\sin(t+3/2)\theta/\sin(\theta/2)$ be a function defined on the open interval $(0, 2\pi)$. Then $R(\theta)$ is a continuous function and $R(\theta) = 1$ has $(2t+2)$ roots in $(0, 2\pi)$. They are $\theta = (2k-1)\pi/(t+2)$ and $\theta = 2h\pi/(t+1)$ where $k = 1, \ldots, t+2$ and $h = 1, \ldots, t$. It is easy to see that by continuity (or the mean value theorem) we can choose a small real number $\varepsilon \neq 0$ so that $R(\theta) = 1 + \varepsilon$ has $2t+2$ roots in $(0, 2\pi)$. The reader can check this by drawing the graph for $y = R(\theta)$. Moreover, for any $s \in (1, 1+\varepsilon)$ (or $s \in (1+\varepsilon, 1)$) $R(\theta) = s$ has $2t+2$ roots in $(0, 2\pi)$. This shows that there are many ($\aleph_1$) non-group-type t-designs in the case $n = 2t+3$. We want to pick a subset of these non-group-type t-designs to construct non-group-type t-designs of bigger size (size $\geq 3t+4$). First, we make the following preparation.

DEFINITION. An angle $\theta \in \mathbb{R}$ is rational if $\theta = q\pi$ for some rational number q. Otherwise, θ is irrational.

PROPOSITION 3. *If θ is a rational angle, then* $\cos\theta$ *and* $\sin\theta$ are algebraic numbers.

COROLLARY 3.1. *If R is a transcendental number, then all the roots θ of $R = -\sin(t+3/2)\theta/\sin(\theta/2)$ are irrational angles.*

From the above discussion, Corollary 3.1 and the fact that there are $\aleph_1$ transcendental numbers sitting in $(1, 1+\varepsilon)$, (or in $(1+\varepsilon, 1)$) we conclude that, in the case $n = 2t+3$, there are $\aleph_1$ non-group-type t-designs $X = \{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_{2t+3}}\}$ with the property that

$\theta_1=0$ and $\theta_k (2 \leqslant k \leqslant 2t+3)$ are distinct irrational angles. This is also true for cases $2t+4 \leqslant n \leqslant 3t+3$; the argument is the same as (at least similar to) above. Thus we have the following lemma.

LEMMA 4. *For each size* $n \in (2t+3, 3t+3)$, *there are* $\aleph_1$ *non-group-type t-designs* $X=\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\}$ *with the property that* $\theta_1=0$ *and* $\theta_k (2 \leqslant k \leqslant n)$ *are distinct irrational angles.*

Using Lemma 4, we can construct many ($\aleph_1$) non-group-type t-designs of size $\geqslant 3t+4$. This is done in the following way.

Given an integer $n \geqslant 3t+4$, let $n=(s+2)(t+1)+k$ where $1 \leqslant k \leqslant t+1$ and $s \geqslant 1$. Take a non-group-type t-design X_0 of size $2(t+1)+k=m$ as described in Lemma 4; i.e. $X_0=\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_m}\}$ where $\theta_1=0$ and $\theta_k (2 \leqslant k \leqslant m)$ are irrational angles. We want to use X_0 to construct a non-group-type t-design Y of size n. Choose a rational angle δ such that $0<\delta<2\pi/sn$. For $1 \leqslant j \leqslant s$, let X_j be the regular $(t+1)$-gon containing e^{ia_j} where $a_j=j\delta$. Let $Y=\bigcup_{i=0}^{s} X_i$. Then Y is a non-group-type t-design of size n. Suppose not. Then Y could be decomposed into the union of some regular k_i-gons where $t+1 \leqslant k_i \leqslant n$. Because of the choice of δ, each $e^{i\theta_1}, e^{ia_1}, \ldots, e^{ia_s}$ would belong to a different k_i-gon in the decomposition. This would give us $\geqslant (t+1)(s+1)>(t+1)s+1$ points $e^{i\theta}$ in Y with rational angles θ, which contradicts the fact that Y has only $(t+1)s+1$ points $e^{i\theta}$ in it with rational angles θ. We notice that:

(a) the above construction of Y depends on the choice of X_0;
(b) different X_0's give different Y's;
(c) For each $n \geqslant 3t+4$, there are $\aleph_1$ such X_0 to choose.

Hence we get the second assertion of Theorem A. That is

For each $n \geqslant 3t+4$, there are $\aleph_1$ non-group-type t-designs of size n.

4. REMARK

It will be very interesting if we can get a similar result to Theorem A for $d \geqslant 3$. In general, we can define a spherical t-design X in $\mathbb{R}^d$ to be a group-type t-design if X is the union of spherical t-designs X_i where each X_i has a finite subgroup of the orthogonal group $O(d)$ acting on X_i transitively. We ask the following questions for $d \geqslant 3$.

(i) When $|X|$ is close to the lower bound as stated in [1, Theorems 5.11 and 5.12], is X necessarily a group-type t-design?
(ii) Can we show the existence of many non-group-type spherical t-designs for large t? So far we do not know any spherical t-designs for large t and for $d \geqslant 3$.

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